

# Evolutionary Game Dynamics: Finite Populations

## A Motivating Example

### Setup

Consider a two-person game. Suppose there are two types of players: type  $A$  and type  $B$ . When a type  $A$  goes against a type  $B$ , the former reaps  $b$  rewards whereas the latter reaps  $c$  rewards. When a type  $A$  goes against another type  $A$ , both reap  $a$  rewards. When a type  $B$  goes against another type  $B$ , both reap  $d$  rewards. We can summarize this payoff information into a payoff matrix:

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let  $\pi_A$  be the proportion of type  $A$  players in the population, and  $\pi_B$  be the proportion of type  $B$  players. Note that  $\pi_A + \pi_B = 1$ . We can define a fitness function  $f : \{A, B\} \rightarrow \mathbb{R}$  such that

$$\begin{aligned} f(A) &= \pi_A \cdot a + \pi_B \cdot b \\ f(B) &= \pi_A \cdot c + \pi_B \cdot d \end{aligned}$$

Note that these fitness functions really do match our intuitions about “fitness”; intuitively, the fitness of type  $A$  should be a weighted average of the rewards from playing a type  $A$  versus type  $B$  player.

Finally, we define the average fitness of a population:

$$\phi = f(A) \cdot \pi_A + f(B) \cdot \pi_B$$

### Exploration

Our main question revolves around how  $\pi_A, \pi_B$  change as time goes on. We present a few questions of interest:

- How “fit” must a type be so that it eventually takes over the population i.e.  $\pi_A = 1$ ?
- If no takeover is guaranteed, at which proportions of  $A, B$  does the population “stabilize”?
- Do the initial proportions of  $A, B$  matter?

To address these questions, we need to introduce dynamics into our static model. We have the following differential equations for how the proportions of  $A, B$  change over time.

$$\begin{aligned} \dot{\pi}_A &= \pi_A(f(A) - \phi) \\ \dot{\pi}_B &= \pi_B(f(B) - \phi) \end{aligned}$$

Consider the intuition behind these differential equations. It stands to reason that if the fitness of type  $A$  is better than the average population fitness, then the proportion of  $A$ 's should grow. The same intuition applies for type  $B$  also.

Now, because  $\pi_A + \pi_B = 1$ , we can summarize the equations above to

$$\begin{aligned} \dot{\pi}_A &= \pi_A(f(A) - f(A) \cdot \pi_A - f(B) \cdot (1 - \pi_A)) \\ &= \pi_A(1 - \pi_A)(f(A) - f(B)) \end{aligned}$$

Furthermore, notice that

$$f(A) - f(B) = \pi_A \cdot (a - c) + (1 - \pi_A) \cdot (b - d)$$

**Fitness Requirements for Takeover** If  $a > c$  and  $b > d$ , then  $A$  will completely dominate  $B$ . This is because  $f(A) - f(B)$  is positive, so  $\pi_A$  is positive, which means that  $\pi_A$  will grow indefinitely. Similarly, if  $a < c$  and  $b < d$ , then  $B$  will completely dominate  $A$  by similar logic.

**Stabilization** To be precise, stabilization asks under what conditions the population proportions stay constant. In other words, when is  $\pi_A = 0$ ?

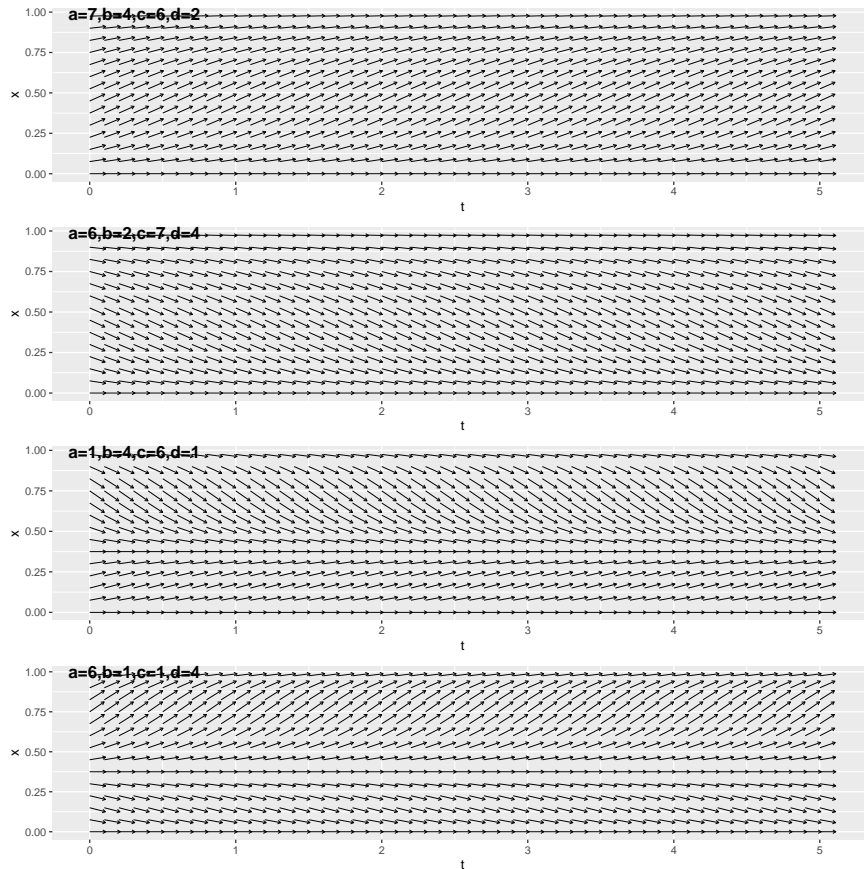
Clearly,  $\pi_A = 0$  when  $\pi_A = 0$  or  $\pi_A = 1$ . Are there any other equilibrium points?

A little bit of work shows that when  $a > c$  and  $d > b$  (or  $a < c$  and  $d < b$ ), there exists a third equilibrium point

$$\pi_A = \frac{d - b}{a - b - c + d}.$$

In dynamics, equilibrium points are classified as either stable or unstable. A stable equilibrium point is one that is “attractive”; nearby points travel towards it. In contrast, an unstable equilibrium point is one that is “repulsive”; nearby points travel away from it.

Below are several plots with different  $a, b, c, d$  values. We have plotted the vector field corresponding to  $\pi_A = \pi(1 - \pi_A)(f(A) - f(B))$ ; the  $x$ -axis represents time, and the  $y$ -axis is represents  $\pi_A$ .



- In the first plot, we see that  $\pi_A = 1$  is a stable equilibrium, but  $\pi_A = 0$  is an unstable equilibrium; this is expected since we see that if  $a > c$  and  $b > d$ , then type  $A$  will takeover eventually.
- In the second plot, we see a similar picture;  $\pi_A = 0$  is a stable equilibrium and  $\pi_A = 1$  is an unstable equilibrium.
- In the third and fourth plots, we notice that there is a third equilibrium point along with  $\pi_A = 0, 1$ :  $\pi_A = \frac{3}{8}$ . This third equilibrium point is a stable equilibrium in the third plot, but is an unstable

equilibrium in the fourth plot.

**Initial Conditions** The plots also show that the initial conditions matter only when  $a > c$  and  $d > b$  for converging onto or away from an equilibrium point. Depending on whether  $\pi_A$  starts, the  $A$  types may either completely take over or completely disappear.

### Takeaway

With a dynamical perspective on this simple two-player game, we were able to extract insights on how the game evolves based purely on the payoffs of each type from battling every other type. However, there is much left to be desired.

In particular, our dynamical approach assumes an infinite population. However, infinite populations are unrealistic, so we should turn our attention to finite populations instead. Unfortunately, finiteness also introduces “noise” into the dynamics. It hence becomes necessary to use probabilistic techniques and stochastic models.

In this way, this simple example leads us to another avenue - viewing evolutionary game dynamics for finite populations through a **stochastic** lens.

## Finite Populations

### Setup

Suppose we have a population of  $N$  individuals. An individual can either be of type  $A$  or type  $B$ . Let  $X_t$  be the number of type  $A$  individuals at time  $t$ , where time is discrete. Neglecting fitness for the time being, let

$$\begin{aligned}\lambda_j &= \Pr\{\text{going from } j \text{ to } j + 1 \text{ Type } A \text{ individuals}\} \\ \mu_j &= \Pr\{\text{going from } j \text{ to } j - 1 \text{ Type } A \text{ individuals}\}.\end{aligned}$$

Colloquially, we refer to  $\lambda$  and  $\mu$  as the birth and death rates, respectively. The observant reader will notice that our setup describes a discrete-time Markov Chain.

We are interested in the following two questions:

1. How likely is it for  $j$  individuals of type  $A$  to take over the population? We call this the *fixation probability*.
2. What is the expected time for  $j$  individuals of type  $A$  to take over the population? We call this the *fixation time*.

### Exploration

**Fixation Probability** Our goal is to calculate

$$f_A(j) = \Pr\{j \text{ individuals of type } A \text{ take over}\}.$$

It is clear that  $f_A(0) = 0$  and  $f_A(N) = 1$ . For  $0 < j < N$ , we have the following recurrence relation:

$$f_A(j) = \mu_j \cdot f_A(j - 1) + \lambda_j \cdot f_A(j + 1) + (1 - \mu_j - \lambda_j) \cdot f_A(j).$$

This is because a type  $A$  population could reduce by one, increase by one, or stay the same. Rearranging this equation, we get the following:

$$\begin{aligned}f_A(j + 1) - f_A(j) &= \frac{\lambda_j}{\mu_j} \cdot (f_A(j) - f_A(j - 1)) \\ &= \frac{\lambda_1 \cdots \lambda_j}{\mu_1 \cdots \mu_j} \cdot f_A(1).\end{aligned}$$

In this way, we have

$$\begin{aligned} f_A(j+1) &= [f_A(j+1) - f_A(j)] + \cdots + [f_A(2) - f_A(1)] + [f_A(1) - f_A(0)] \\ &= f_A(1) + \sum_{k=1}^j \frac{\lambda_1 \cdots \lambda_k}{\mu_1 \cdots \mu_k} \cdot f_A(1) \end{aligned}$$

Hence, finding  $f_A(j+1)$  reduces to finding  $f_A(1)$ . We know that  $f_A(N) = 1$ , so we get that

$$1 = f_A(N) = f_A(1) + \sum_{k=1}^{N-1} \frac{\lambda_1 \cdots \lambda_k}{\mu_1 \cdots \mu_k} \cdot f_A(1) \implies f_A(1) = \left( 1 + \sum_{k=1}^{N-1} \frac{\lambda_1 \cdots \lambda_k}{\mu_1 \cdots \mu_k} \right)^{-1}.$$

In contrast, if we denote  $f_B(j) = \Pr\{j \text{ individuals of type } B \text{ take over}\}$ , then we see that

$$\begin{aligned} f_B(1) &= 1 - f_A(N-1) \\ &= \frac{\sum_{k=1}^{N-1} \frac{\lambda_1 \cdots \lambda_k}{\mu_1 \cdots \mu_k}}{1 + \sum_{k=1}^{N-1} \frac{\lambda_1 \cdots \lambda_k}{\mu_1 \cdots \mu_k}} - \sum_{k=1}^{N-2} \frac{\lambda_1 \cdots \lambda_k}{\mu_1 \cdots \mu_k} \cdot f_A(1). \end{aligned}$$

Using  $f_B(1)$ , we can find  $f_B(j)$  using the same logic as we did for finding  $f_A(j)$ .

**Fixation Time** Notice that there are two absorbing states: when type  $A$  takes over or when type  $A$  becomes extinct. We define the following random variable  $Y$

$$\begin{aligned} Y &= \text{time to reach an absorbing state} \\ &= \min\{t : X_t = 0 \text{ or } X_t = N\}. \end{aligned}$$

Then for  $0 < j < N$ , we then want to find

$$T_A(j) = \mathbb{E}[Y | X_0 = j].$$

Notice again that  $T_A(0) = T_A(N) = 0$ . We have the following recurrence relation:

$$T_A(j) = 1 + \mu_j \cdot T_A(j-1) + \lambda_j \cdot T_A(j+1) + (1 - \lambda_j - \mu_j) \cdot T_A(j).$$

The 1 in the recurrence relation comes from the fact that it takes one time step to change state. Now, rearranging this equation, we get the following:

$$(\lambda_j + \mu_j) \cdot T_A(j) = 1 + \mu_j \cdot T_A(j-1) + \lambda_j \cdot T_A(j+1).$$

Define a matrix  $A \in \mathbb{R}^{N+1 \times N+1}$  such that

$$A_{ij} = \begin{cases} -(\lambda_j + \mu_j) & \text{if } j = i \\ \mu_j & \text{if } j = i - 1 \\ \lambda_j & \text{if } j = i + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then let  $\tilde{A}$  be the sub-matrix of  $A$  obtained by deleting the the first and  $N + 1$ th rows, and the first and  $N + 1$ th columns. In other words,  $\tilde{A}$  is  $A$  but without the rows and columns associated with 0 and  $N + 1$  Type  $A$  individuals. The observant reader will recognize the resemblance between  $A$  and infinitesimal generators.

We can then express our equation as

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + \tilde{A} \begin{bmatrix} T_A(1) \\ \vdots \\ T_A(N-1) \end{bmatrix}.$$

Rearranging, we get that

$$\boxed{\begin{bmatrix} T_A(1) \\ \vdots \\ T_A(N-1) \end{bmatrix} = [-\tilde{A}]^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}}.$$

In fact, this method can be extended. We can actually find the expected time to reach *any* state  $z$  by simply deleting the rows and columns associated with  $z$  from  $\tilde{A}$  and carrying out the same inverse operations.

*Note: For those that are curious about the invertibility of  $\tilde{A}$ , we know that  $\tilde{A}$  is a square matrix with nonpositive row sums and at least one row sum strictly negative. This means that  $\tilde{A}$  has eigenvalues with all strictly negative real part, so  $\tilde{A}$  is invertible.*

## Moran Processes & Rock-Paper-Scissors (RPS)

### Example

Suppose  $N = 3$  and  $\lambda_j = \mu_j = 0.2$  for  $j = 1, 2$ . The possible states for the number of possible number of type  $A$  individuals are then 0, 1, 2, 3. Then it follows that

$$f_A(1) = \left( 1 + \sum_{k=1}^2 \frac{\lambda_1 \cdots \lambda_k}{\mu_1 \cdots \mu_k} \right)^{-1} = \frac{1}{3}.$$

Furthermore, our  $A$  matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ \mu_1 & -(\mu_1 + \lambda_1) & \lambda_1 & 0 \\ 0 & \mu_2 & -(\mu_2 + \lambda_2) & \lambda_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{ so } \tilde{A} = \begin{bmatrix} -(\mu_1 + \lambda_1) & \lambda_1 \\ \mu_2 & -(\mu_2 + \lambda_2) \end{bmatrix}.$$

Let  $T = [T_A(1), T_A(2)]^T$ . To check, we recover our original recurrence relations from  $\tilde{A}T$ :

$$\begin{aligned} (\mu_1 + \lambda_1) \cdot T_A(1) &= 1 + \lambda_1 \cdot T_A(2) \\ (\mu_2 + \lambda_2) \cdot T_A(2) &= 1 + \mu_2 \cdot T_A(1). \end{aligned}$$

*Note: these recurrence relations look different to the ones above. However, recall that  $T_A(0) = T_A(3) = 0$ .*

Plugging in our actual birth and death rates, we get

$$\tilde{A} = \begin{bmatrix} -0.4 & 0.2 \\ 0.2 & -0.4 \end{bmatrix} \implies [-\tilde{A}]^{-1} = \begin{bmatrix} \frac{10}{3} & \frac{5}{3} \\ \frac{5}{3} & \frac{10}{3} \end{bmatrix}.$$

Hence, we get that  $T_A(1) = T_A(2) = \frac{10}{3} + \frac{5}{3} = 5$ . It makes sense that both of these expected times are the same because the probabilities of birth and death are equivalent. Therefore, the time it takes to reach either absorbing state should be the same for both states 1 and 2.

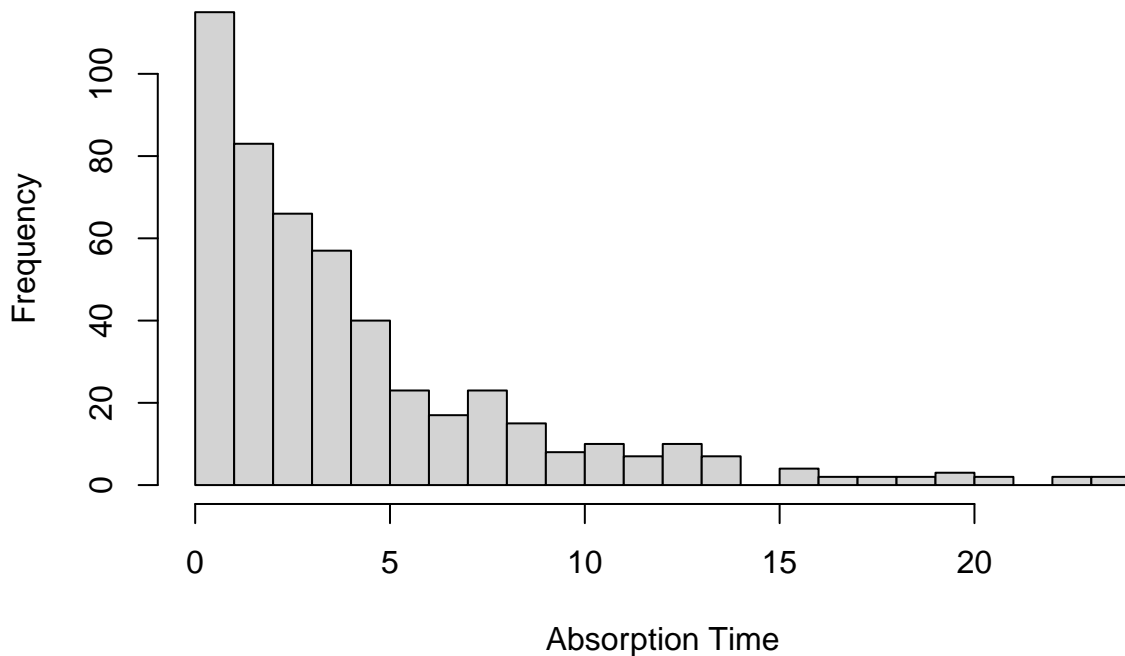
We can test these theoretical results by simulating a Markov chains with the appropriate birth and death rates. Because 0 and 3 are absorbing states, our probability transition matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 0.6 & 0.2 & 0 \\ 0 & 0.2 & 0.6 & 0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

## Average absorption time: 4.706

## Takeover probability: 0.7

### Histogram of Absorption Times



As expected, our empirical results roughly match our theoretical expectations.

#### Takeaway

Although finiteness was initially a big hiccup, we leveraged probabilistic machinery (namely Markov chains) to unveil fixation probabilities and fixation times - two pieces of information that we might be interested in when studying a real game/population.

#### Moran Processes

The Moran process is the model example of a stochastic process that models finite populations. Given that every population is a finite population, the Moran process can potentially apply to every biological population with changing population makeup.

To give a conceptual overview, the Moran process models a finite population by randomly choosing two individuals: one individual to produce an offspring, and another individual to die. The probabilities at which these two individuals are chosen depend on a variety of factors: fitness, selection intensity, etc.

In this vignette, we introduce the frequency-dependent Moran process. The frequency-dependent Moran process assigns probabilities depending on how many of each type of individual currently exists in the

population. More specifically, an individual of a certain type reproduces proportionally to its fitness, which depends on the sub-population of its type.

**Setup** Let  $N$  be the size of the population, and let there be two types of individuals: type  $A$  and type  $B$ . Like all our games, we begin with the payoff matrix:

$$\text{payoff matrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We will leverage Markov chains, where the number of type  $A$  individuals is the state space.

Given that there are  $i$  individuals of type  $A$ , we denote the current fitness of the type  $A$  sub-population as  $f_A(i)$ . Similarly, given that there are  $i$  individuals of type  $A$ , we denote the current fitness of the type  $B$  sub-population as  $f_B(i)$ . With a parameter  $w \in [0, 1]$ , we define

$$\begin{aligned} f_A(i) &= 1 - w + w \left[ a \cdot \frac{i-1}{N-1} + b \cdot \frac{N-i}{N-1} \right] \\ f_B(i) &= 1 - w + w \left[ c \cdot \frac{i}{N-1} + d \cdot \frac{N-i-1}{N-1} \right]. \end{aligned}$$

By varying  $w$ , we see that  $w$  models the selection intensity. Indeed, when  $w$  is close to 1, then we have strong selection since the phenotypes of either type make *more* of a difference in fitness. In contrast, when  $w$  is close to 0, then we have weak selection since the phenotypes of either type make *less* of a difference in fitness.

We denote the average fitness of the population as

$$\phi(i) = f_A(i) \cdot \frac{i}{N} + f_B(i) \cdot \frac{N-i}{N}.$$

Now, we said previously that an individual of a certain type reproduces proportionally to its fitness. To this end, we set the probability of a type  $A$  individual reproducing as

$$\frac{f_A(i)}{\phi(i)} \cdot \frac{i}{N}.$$

For death, we assign death probabilities uniformly. Therefore, to go from  $i$  to  $i+1$  type  $A$  individuals, we must pick an  $A$  individual to reproduce and a  $B$  individual to die. The probability of going from  $i$  to  $i+1$  type  $A$  individuals is hence

$$\lambda_i = \underbrace{\frac{f_A(i)}{\phi(i)} \cdot \frac{i}{N}}_{\text{reproduction}} \cdot \underbrace{\frac{N-i}{N}}_{\text{death}}.$$

Similarly, the probability of going from  $i$  to  $i-1$  individuals is

$$\mu_i = \underbrace{\frac{f_B(i)}{\phi(i)} \cdot \frac{N-i}{N}}_{\text{reproduction}} \cdot \underbrace{\frac{i}{N}}_{\text{death}}.$$

This then defines a birth-death Markov chain - just as shown in the previous vignette. We have the following transition probabilities, with  $i$  being the number of type  $A$  individuals:

$$\begin{aligned} \Pr\{i \rightarrow i+1\} &= \frac{f_A(i)}{\phi(i)} \cdot \frac{i(N-i)}{N^2} \\ \Pr\{i \rightarrow i-1\} &= \frac{f_B(i)}{\phi(i)} \cdot \frac{i(N-i)}{N^2} \\ \Pr\{i \rightarrow i\} &= 1 - \lambda_i - \mu_i \end{aligned}$$

With this, we can calculate the fixation probabilities and fixation times.

**Fixation Probabilities** Our aim is to calculate the probability of type  $A$  individuals taking over, given that there are initially  $j$  individuals.

To do this, recall that we must calculate

$$\pi_A(1) = \Pr\{1 \text{ type } A \text{ individual taking over}\} = \left(1 + \sum_{k=1}^{N-1} \frac{\lambda_1 \cdots \lambda_k}{\mu_1 \cdots \mu_k}\right)^{-1}.$$

It will be helpful to calculate  $\frac{\lambda_i}{\mu_i}$ , where  $1 \leq i \leq N-1$ . We have

$$\frac{\lambda_i}{\mu_i} = \frac{\frac{f_A(i) \cdot i(N-i)}{\phi(i) \cdot N^2}}{\frac{f_B(i) \cdot i(N-i)}{\phi(i) \cdot N^2}} = \frac{f_A(i)}{f_B(i)}.$$

This means that

$$\pi_A(1) = \left(1 + \sum_{k=1}^{N-1} \frac{f_A(1) \cdots f_A(k)}{f_B(1) \cdots f_B(k)}\right)^{-1}.$$

**Fixation Times** In this case, our  $A$  matrix is

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \mu_1 & -(\mu_1 + \lambda_1) & \lambda_1 & \cdots & 0 \\ \vdots & \mu_2 & -(\mu_2 + \lambda_2) & \ddots & 0 \\ \vdots & \vdots & \vdots & -(\mu_{N-1} + \lambda_{N-1}) & \lambda_{N-1} \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

From this, we get our transition times

$$- \begin{bmatrix} -(\mu_1 + \lambda_1) & \lambda_1 & \cdots & 0 \\ \mu_2 & -(\mu_2 + \lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -(\mu_{N-1} + \lambda_{N-1}) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

We now move on to extending the ideas of the Moran process to Rock, Paper, Scissors.

### Rock, Paper, Scissors (RPS)

So far, we have only considered two-type games. Using the classic game RPS as a guide, we will extend the two-type Moran process to handle three types.

While silly from the outset, RPS is actually applicable to biology. In particular, RPS can serve as a good model to explain biodiversity within a small area. Notice that the rules of RPS dictate a kind of “cyclical” dominance: paper beats rock, rock beats scissors, scissors beats paper, paper beats rock, *ad infinitum*. These cyclical rules potentially explain how many species can coexist in the same region; species can cycle in and out of dominance, ultimately allowing for coexistence.

**Setup** Let  $N$  be the size of the population, and let there be three types of individuals:  $R$  (rock),  $P$  (paper), and  $S$  (scissor).

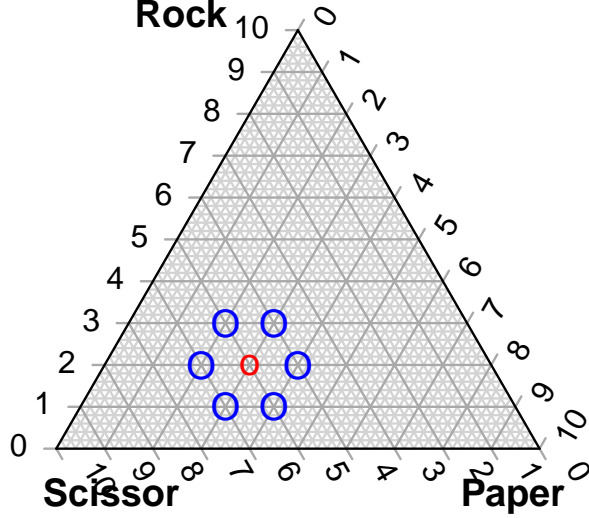
For a parameter  $s > 0$ , the RPS payoff matrix is as follows:

$$\text{payoff matrix} = \begin{bmatrix} 0 & -s & 1 \\ 1 & 0 & -s \\ -s & 1 & 0 \end{bmatrix}$$



It is clear from the payoff matrix here that paper beats rock, rock beats scissors, and scissors beats paper. The winner gains 1 unit, whereas the loser loses  $s$  units (hence the negative sign on  $s$ ).

Now, since there are three types of players in RPS, it is useful to think about RPS as a Moran-esque game on a *simplex*. Define a simplex  $S_N = \{(i, j, k) : i + j + k = N\}$ . We have illustrated a simplex  $S_{10}$  with  $N = 10$  below.



Suppose we start out at  $(2, 3, 5)$  i.e. 2 Rock's, 3 Paper's, and 5 Scissor's, denoted by the red point. Then we randomly choose two individuals: an individual to reproduce and an individual to die. This is equivalent to moving to one of the blue circles, or staying in the same place. For example, if we choose  $R$  to reproduce and  $S$  to die, then we move from the red point to the blue point at the top-right of the hexagon. Similarly, if we choose  $P$  to reproduce and  $P$  to die, then we stay at the red point. In other words, RPS represents a **random walk** on the simplex.

Using this intuition, we deduce that a two-dimensional state space must be used for our Markov chain representation, with the first dimension denoting the number of Rock individuals and the second dimension denoting the number of Paper individuals. (Since we are dealing with a finite population, the number of Scissor individuals can be ascertained completely from the number of Rock and Paper individuals.)

Given that there are  $(i, j)$  individuals of type  $R$  and  $P$  respectively, define  $f_R(i, j)$ ,  $f_P(i, j)$ , and  $f_S(i, j)$  as follows:

$$f_R(i, j) = \frac{N - i - j}{N - 1} + \frac{j}{N - 1} \cdot (-s)$$

$$f_P(i, j) = \frac{i}{N - 1} + \frac{N - i - j}{N - 1} \cdot (-s)$$

$$f_S(i, j) = \frac{j}{N - 1} + \frac{i}{N - 1} \cdot (-s).$$

We denote the average fitness of the population as

$$\phi(i, j) = f_R(i, j) \cdot \frac{i}{N} + f_P(i, j) \cdot \frac{j}{N} + f_S(i, j) \cdot \frac{N - i - j}{N}.$$

For choosing the reproducing individual, we introduce four reproductive functions. Reproductive functions define the fitness of the reproducing type, parametrized by the dying type. When combined with the frequencies of the reproducing and dying types, a reproductive function expresses the transition probability completely. With  $b$  as the reproducing individual,  $a$  as the dying individual, and a parameter  $w \in [0, 1]$ , we

have

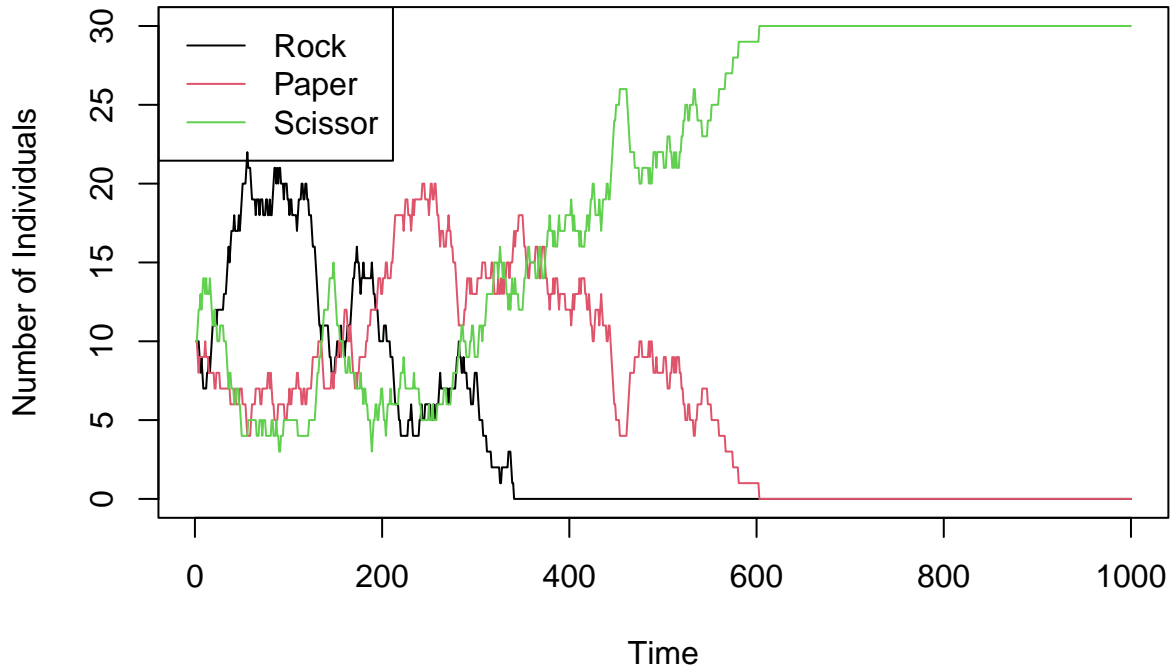
$$\begin{aligned}\Phi_{MO}(b \rightarrow a) &= \frac{1 - w + w \cdot f_a(i, j)}{1 - w + w \cdot \phi(i, j)} \text{ (Moran)} \\ \Phi_{LM}(b \rightarrow a) &= 1 + w \cdot (f_a(i, j) - \phi(i, j)) \text{ (Linear Moran)} \\ \Phi_{LU}(b \rightarrow a) &= 1 + w \cdot (f_a(i, j) - f_b(i, j)) \text{ (Local Update)} \\ \Phi_{FP}(b \rightarrow a) &= [1 + \exp(-w \cdot (f_a(i, j) - f_b(i, j)))]^{-1} \text{ (Fermi Process)}\end{aligned}$$

Since we choose the death types uniformly, we have the following transition probabilities:

$$\begin{aligned}\Pr\{(i, j) \rightarrow (i + 1, j - 1)\} &= \Phi(R \rightarrow P) \cdot \frac{ij}{N^2} \\ \Pr\{(i, j) \rightarrow (i - 1, j + 1)\} &= \Phi(P \rightarrow R) \cdot \frac{ij}{N^2} \\ \Pr\{(i, j) \rightarrow (i + 1, j)\} &= \Phi(R \rightarrow S) \cdot \frac{i(N - i - j)}{N^2} \\ &\vdots\end{aligned}$$

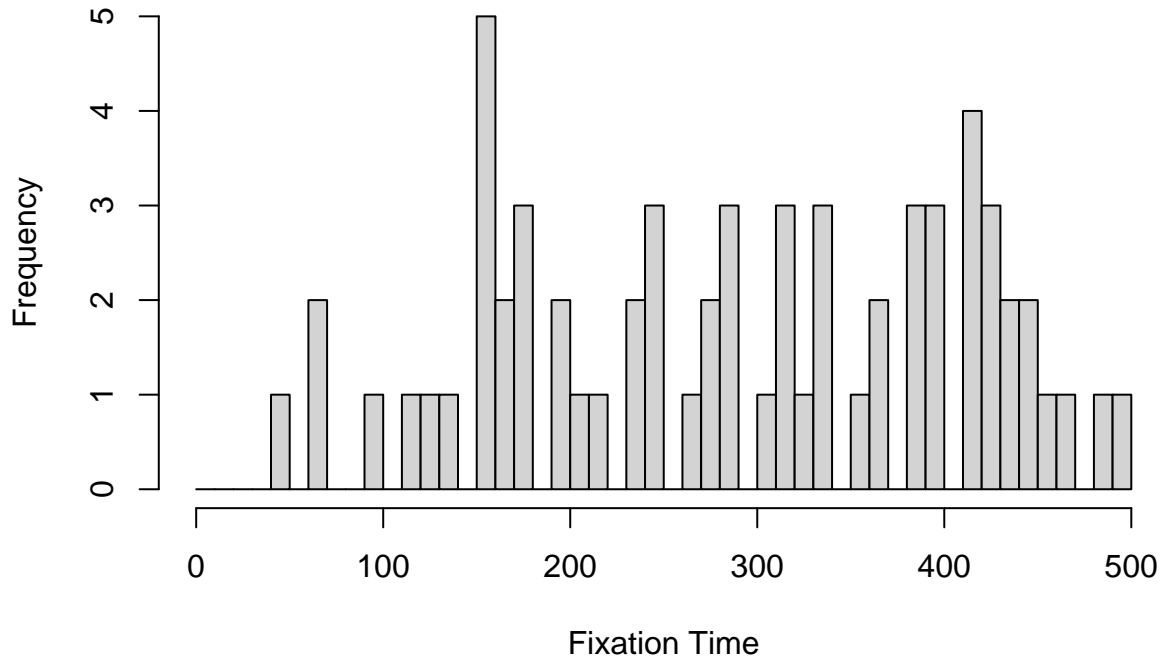
**Fixation** In RPS, the absorption states are  $(0, 0)$ ,  $(0, N)$ ,  $(N, 0)$ . Clearly, these three states are states in which one type has completely taken over the population.

Unfortunately, determining the fixation probabilities and fixation times of RPS is difficult. This is because we are no longer in a strict birth-death situation given the two-dimensional state space. However, we can try to see if there exists any expected fixation time at all.



Here, we have simulated a game of RPS with 30 individuals for 1000 time steps. The selection intensity is 0.1. We ran 100 RPS games for 500 steps, but unfortunately cannot find any pattern in the fixation time.

## Histogram of Fixation Times



### Takeaway

We introduced Moran processes as a standard stochastic process for modeling finite populations. We further extended Moran processes to the classic game Rock, Paper, Scissors (RPS). However, we were not able to deduce the exact formulas for fixation probability and fixation time. Furthermore, the empirical evidence suggests that no fixation time exists in the first place.

Given that the principles of RPS can be further extended to more complicated games, the reader will be able to use the ideas in this vignette for more complex research problems.

To close, we run through a brief summary. Our first vignette introduced a model for evolutionary game dynamics of infinite populations using ordinary differential equations. However, because infiniteness is an unrealistic assumption, we introduced finiteness. Using probabilistic machinery such as Markov chains, we laid down a basic structure for finite population game modeling in the second vignette. Finally, this third vignette utilized the basic structure for Moran processes and RPS.

### References

1. Traulsen, Arne, and Christoph Hauert. "Stochastic evolutionary game dynamics." *Reviews of nonlinear dynamics and complexity* 2 (2009): 25-61.
2. Ferreira, Eliza M., and Armando GM Neves. "Fixation probabilities for the Moran process with three or more strategies: general and coupling results." *Journal of Mathematical Biology* 81, no. 1 (2020): 277-314.
3. Taylor, Christine, Drew Fudenberg, Akira Sasaki, and Martin A. Nowak. "Evolutionary game dynamics in finite populations." *Bulletin of mathematical biology* 66, no. 6 (2004): 1621-1644.
4. Yu, Qian, Debin Fang, Xiaoling Zhang, Chen Jin, and Qiyu Ren. "Stochastic evolution dynamic of the rock-scissors-paper game based on a quasi birth and death process." *Scientific Reports* 6, no. 1 (2016): 1-9.